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PHASE SHIFTS FROM GABOR FILTERS

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A technique is presented for extracting phase information from a pair of sinusoidal signals, with equal frequency, but different phase angle, by convolving each signal with a pair of quadrature Gabor filters.

It is shown that, under certain conditions, Gabor filters can be treated as linear functions. The applications of the Gabor filter are discussed with respect to both image processing, and extracting phase information from sinusoidal signals in minimum time.

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Phase shifts from Gabor filters

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ABSTRACT

A technique is presented for extracting phase information from a set of Gabor filters. The results of this work, which form a preliminary investigation into the properties of Gabor Filters in Image Processing indicates precise conditions from which the filters may be considered linear from phase analysis.

1. Introduction

The Gabor[1] signal, which provides a concise representation of a signal in both spatial and spatial frequency domains, has been subject to considerable attention in many areas of signal processing.

Indeed, as a direct consequence of the scaling theorem $af(ax) \rightarrow F(\omega/a)$, Gabor formulated his famous Uncertainty Principle, which states that a signal cannot be located arbitrarily in both the signal domain, and signal frequency domain. He found that within the Uncertainty Principle, there existed a set of functions which minimise the uncertainty between the signal and frequency domain.

Marcjela[2] was the first to recognise the similarity between the receptive filed profiles of simple cells in the mammalian striate cortex, and the even or odd elementary signals of Gabor. The use of the Gabor function to simulate biological vision processes has been formulated by several authors.[3, 4]

However, these authors limited their analysis to the spectral energy response from the expansion of a signal into it's elementary signals. Larcombe, (personal communication) and subsequently Wilson[5] have suggested that phase information, may well have important properties in estimating stereoscopic disparities between a pair of images. This paper therefore provides a formulation for extracting phase information from a set of quadrature Gabor filters.

Let us consider two sinusoidal signals, with equal frequency but a spatial phase difference. By obtaining an estimate of both the spatial-frequency and the phase difference between the signals it is possible to compute the displacement of the second signal.

Consider applying a pair of quadrature Gabor filters, at the same spatial location to both sinusoid signals.

$$I_s = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \exp[\frac{-(x-X)^2}{4\sigma^2}] \cos 2\pi u_g (x-X) F(X)\, dX \tag{1}$$

$$I_a = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \exp[\frac{-(x-X)^2}{4\sigma^2}] \sin 2\pi u_g (x-X) F(X)\, dX \tag{2}$$

$$I_{s\phi} = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \exp[\frac{-(x-X)^2}{4\sigma^2}] \cos 2\pi u_g (x-X) F(X+\Phi)\, dX \tag{3}$$

$$I_{a\phi} = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \exp[\frac{-(x-X)^2}{4\sigma^2}] \sin 2\pi u_g (x-X) F(X+\Phi)\, dX \tag{4}$$

If we let $F(X)=\cos 2\pi u_o$ and expanding equations (3) and (4) , we observe that;

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$$I_{s\phi} = Cos\,\phi\, I_s - Sin\,\phi\, I_1 \tag{5}$$

Where I_1 is;

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \exp[\frac{-(x-X)^2}{4\sigma^2}] \cos 2\pi u_g (x-X) \sin 2\pi u_o X \; dX \tag{6}$$

and

$$I_{a\phi} = \sin\phi\, I_2 - \cos\phi\, I_a \tag{7}$$

where;

$$I_2 = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \exp[\frac{-(x-X)^2}{4\sigma^2}] \sin 2\pi u_g (x-X) \sin 2\pi u_o X \; dX \tag{8}$$

By taking the Fourier transform of I_s and observing that $F(x) * G(x) \rightarrow F(u) G(u)$ we arrive at the following equation;

$$F(u) = (8\pi\sigma^2)^{\frac{1}{4}} (\exp[-4\pi^2\sigma^2(u-u_g)^2] + \exp[-4\pi^2\sigma^2(u+u_g)^2]) \frac{1}{2}(\delta(u-u_o)+\delta(u+u_o)) \tag{9}$$

The inverse Fourier transform of the above equation effectively provides a solution to the integral in (1);

$$I_s = (8\pi\sigma^2)^{\frac{1}{4}} (\exp[-4\pi^2\sigma^2(u_o-u_g)^2] + \exp[-4\pi^2\sigma^2(u_o+u_g)^2]) \cos 2\pi u_o x \tag{10}$$

By applying the same technique to equations (2),(6) and (8) then we arrive at the following expressions;

$$I_a = (8\pi\sigma^2)^{\frac{1}{4}} (\exp[-4\pi^2\sigma^2(u_o-u_g)^2] - \exp[-4\pi^2\sigma^2(u_o+u_g)^2]) \sin 2\pi u_o x \tag{11}$$

$$I_1 = (8\pi\sigma^2)^{\frac{1}{4}} (\exp[-4\pi^2\sigma^2(u_o-u_g)^2] + \exp[-4\pi^2\sigma^2(u_o+u_g)^2]) \sin 2\pi u_o x \tag{12}$$

$$I_2 = (8\pi\sigma^2)^{\frac{1}{4}} (-\exp[-4\pi^2\sigma^2(u_o-u_g)^2] + \exp[-4\pi^2\sigma^2(u_o+u_g)^2]) \cos 2\pi u_o x \tag{13}$$

By forming the simple ratio $\frac{I_1}{I_a}$ and simplifying then;

$$\frac{I_1}{I_a} = \frac{\exp[8\pi^2\sigma^2 u_o u_g] + \exp[-8\pi^2\sigma^2 u_o u_g]}{\exp[8\pi^2\sigma^2 u_o u_g] - \exp[-8\pi^2\sigma^2 u_o u_g]} \tag{14}$$

i.e

$$I_1 = \frac{I_a}{\tanh(8\pi^2\sigma^2 u_o u_g)} \tag{15}$$

Using a similar method it can also be shown that;

$$I_2 = -I_s \tanh(8\pi^2\sigma^2 u_o u_g) \tag{16}$$

We may now substitute equations (13) and (14) into equations (5) and (7) from which the phase may be represented in terms of a normal rotation matrix $\underline{R}_{\theta\zeta}$ where $\zeta = \tanh(8\pi^2\sigma^2 u_o u_g)$ i.e

$$\begin{bmatrix} I_{s\phi} \\ I_{a\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\frac{\phi}{\zeta} \\ \sin\phi\ \zeta & \cos\phi \end{bmatrix} \begin{bmatrix} I_s \\ I_a \end{bmatrix} \tag{17}$$

Clearly, if $\zeta = 1.0$ then the Gabor filter may be considered linear in terms of phase analysis. Therefore, if $u_o = u_g$ then $\zeta = 1.0$ providing $u_g = \frac{1}{\pi\sigma}$.

Granlund,[6] shows in principle how Gabor filters may be used to provide uniform coverage of the frequency spectrum by applying filters with different centre frequencies and bandwidth. The expansion of a signal into the "elementary signals" of Gabor has been considered by several authors[1, 7] current research will investigate the properties of phase analysis using the Gabor expansion in both Control Engineering and Image Processing.

2. Conclusion

A method has been presented to extract phase information from a set of Gabor filters convolved with two sinusoid signals separated with some arbitrary phase difference. As a consequence of the analysis, it has been shown that Gabor filters may be regarded as linear provided $u_g = \frac{1}{\pi\sigma}$, and the signal and centre frequency of the Gabor filter are equal. The application of this work in terms of image processing is currently under investigation, the results of which will be presented in the near future.

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